A FUNDAMENTAL BIJECTION THEOREM IN SET THEORY

MICHAEL HUGH KNOWLES

Abstract. A new fundamental theorem on bijections in set theory is presented: if the pre-image and image sets of a bijection have an element in common, then one can construct a bijection from the pre-image set with the common element removed onto the image set with the common element removed. Applied to the defining bijections of Dedekind infinite sets, this theorem suggests a paradox that resembles the Banach-Tarski Paradox, except that it is perhaps "too paradoxical": "paradoxical bijections" from non-empty sets onto the empty set. This re-raises important questions about set theory's foundational concepts, especially that of Dedekind-infinite sets. These questions may eventually lead to the formal source of both paradoxes.

1. Introduction

The concept of strict one-to-one correspondences between the elements of two sets, modernly called bijections, is fundamental to every modern variant of Cantor's set theory. Cantor, et al, however, curiously overlooked an elementary yet fundamental theorem that bears on this concept.

2. An elementary "generalized permutation" of a bijection

The theorem to be presented here concerns bijections where the pre-image and image sets have at least one element in common. We will note in passing that today's standard concept of a permutation as the ordering of the elements of a set or the alteration of such an ordering, currently formalized either as an arbitrary bijection from a set onto itself or as the transformation of one such bijection to another, can easily and fruitfully be generalized, extended, and applied to bijections in general. Simple variants of these generalizations/extensions will be used in the proof of this theorem. However, even a basic study of these is beyond the scope of this paper.

Theorem 1. *Given a bijection* $B(SP, SI)$ *from a pre-image set* SP *onto an image set* SI *, where* SP *and* SI *have an element* EC *in common, then using only simple bijectivity preserving operations one can construct a bijection* B' *from* $SP - \{EC\}$ *onto* $SI - \{EC\}$ *, i.e.* $B'(SP - \{EC\}, SI - \{EC\})$ *.*

Proof. We have 2 cases possible for the common element EC :

1) If the common element EC is subbijected onto itself under $B(SP, SI)$, then we can entirely remove from B this identity subbijection of the pre-image EC onto its image self, and what remains will trivially be a (sub-) bijection from

Date: Revised December 6, 2011.

2000 *Mathematics Subject Classification.* Primary 03E; Secondary 03E99, 03A05, 00A30.

Key words and phrases. bijection, permutation, set theory, Dedekind infinite, Banach-Tarski Paradox, paradoxical bijection.

$SP-\{EC\}$ onto $SI-\{EC\}$, our needed $B'(SP-\{EC\},SI-\{EC\})$. Bijectivity is trivially preserved by this operation. (The removal of a subbijection is an example of generalizing and extending the standard concept of "permutation" as the bijection of a set onto itself to more general bijectivity preserving operations on bijections.) In particular, we need not "reorder" (a la Cantor) any elements of SP or $SP-\{EC\}$ with respect to SI or $SI-\{EC\}$.

$$\boxed{\begin{array}{ccccc} EP_1 & \underline{EC} & EP_3 & EP_4 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ EI_1 & \underline{EC} & EI_3 & EI_4 & \ldots \end{array}} \Rrightarrow \boxed{\begin{array}{cccc} EP_1 & EP_3 & EP_4 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ EI_1 & EI_3 & EI_4 & \ldots \end{array}}$$

2) If EC is not subbijected onto itself under $B(SP,SI)$, then EC in SP must be subbijected onto some element EI in SI and some element EP in SP must be subbijected onto EC in SI . In a (trivially) bijectivity preserving fashion, we can switch the pre-image elements EC and EP (a standard permutation, thought of as an operation, except that the pre-image and image sets are here not in general the same).

$$\boxed{\begin{array}{ccccc} EP_1 & \overrightarrow{EP} & EP_3 & \overleftarrow{EC} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ EI_1 & \underline{EC} & EI_3 & \underline{EI} & \ldots \end{array}} \Rrightarrow \boxed{\begin{array}{ccccc} EP_1 & \underline{EC} & EP_3 & \underline{EP} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ EI_1 & \underline{EC} & EI_3 & \underline{EI} & \ldots \end{array}}$$

We now have a generalized and bijectivity preserving permutation of B , now with the common element EC identity subbijected onto itself, the pre-image EP subbijected onto the image EI , and the rest of the bijective sub-mapping from $SP-\{EC,EP\}$ onto $SI-\{EC,EI\}$ remaining the same as before. As in case 1), the identity subbijection from EC onto itself can be removed, leaving the needed (sub-) bijection from $SP-\{EC\}$ onto $SI-\{EC\}$, our $B'(SP-\{EC\},SI-\{EC\})$. □

This theorem based operation cannot "generalizedly permute" a bijection into a non-bijective mapping. We should also note the corollary, that if one derives/obtains a non-bijective mapping as a result of applying this operation—or a sequence of such operations—the initial mapping cannot have been a validly bijective mapping.

3. Summary

The beginnings of set theory were filled with controversy, and set theory has had many eminent and preeminent critics/opponents, such as Kronecker, Poincaré, Brouwer, and many others. Even Gauss had earlier passed judgment condemning the use of "completed infinities" which are today accepted as foundational in set theory. Given the often vitriolic and even ad hominem nature of the early criticisms aimed at set theory and at its founder, Georg Cantor, it becomes a serious historical question how so many eminent and preeminent mathematicians, proponents and opponents alike, could have overlooked the extraordinarily simple yet theoretically fundamental Theorem 1, especially as it at least starts to call into critical question the set theoretically foundational concept modernly called "Dedekind-infinite".

Dedekind's concept of "Dedekind-infinite" sets, i.e. of transfinite sets that can be bijected onto proper subsets of themselves, rather adroitly formally summarized the known paradoxes of infinity. However, Dedekind's concept per se neither vetted itself for theoretical soundness, nor helped resolve those paradoxes, let alone put them on a sound theoretical foundation. This vetting and resolution were among the highest priorities Dedekind, Cantor, Hilbert, and other set theory proponents faced with regard to Dedekind's concept, and somehow neglected. They should have used their new formalizations of "sets" and "strict one-to-one correspondences" to more critically examine Dedekind's concept for the inherent inconsistency that might accompany infinity's "inherent" paradoxes, and for other serious problems, *before* making that concept, and thus those paradoxes, essential to the foundations of set theory. Many mathematicians and philosophers were worried at that time that incorporating infinity and its "inherent" paradoxes into mathematics would make the foundations of mathematics inconsistent, even if at first invisibly.

Theorem 1 is not only elementary as theorems go, it's proof is all but trivial, and it is important enough to be considered fundamental in set theory regarding the sine qua non concept of bijections. All who have kindly reviewed it and commented have allowed that it is true in set theory, its proof is sound, and it has never been published. However, they (almost) all expressed theoretical doubts about applying it to Dedekind infinite sets, or rather to the defining bijections from those Dedekind infinite sets onto proper subsets of themselves. We can call such defining bijections "Dedekind infinite bijections" (a non-standard term). The feeling was that the "paradoxical bijections" suggested by naively applying Theorem 1 to Dedekind-infinite bijections would be considered "too paradoxical" compared to the accepted Banach-Tarski Paradox.

A naively straightforward application of Theorem 1 to Dedekind infinite bijections seems to yield yet another paradox in set theory, "paradoxical bijections", a paradox that resembles the Banach-Tarski Paradox, which itself reraised controversy about the consistency of set theory in 1924. Naively applying Theorem 1 to remove all the common elements from the pre-image and image sets of a Dedekind-infinite bijection, one seems to obtain "paradoxical bijections" from non-empty sets onto the empty set, certainly a paradox on a par with the Banach-Tarski Paradox (to which it is likely to turn out to be formally related). The Banach-Tarski Paradox went on to become not only world famous, but accepted as sound formal theory in set theoretic geometry. But "paradoxical bijections" from non-empty sets onto the empty set have seemed to many reviewers to be "too paradoxical" when compared with the accepted Banach-Tarski Paradox.

Theorem 1 may eventually stimulate controversy similar to that which first surrounded the Banach-Tarski Paradox in 1924. But this heretofore overlooked fundamental theorem for bijections still deserves the community attention it should have gotten in Cantor's day, even though the controversy it may raise might seem anachronistic or passé to many.

Retired, Rua Miradouro, 45, Sion, Belo Horizonte, MG, Brazil 30310-640
E-mail address: mhk@mhknowles.net
URL: http://www.mhknowles.net/