Opening a Community Deconstruction of Set Theory

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Abstract

A deconstruction (think of it as a careful forensic analysis, as if CSI was investigating) of set theory is proposed, to be undertaken by the mathematical and philosophical communities. The point of departure for this deconstruction, initiated here, is the concept of Dedekind-infinite, named after its conceiver, the German mathematician Richard Dedekind (1831-1916). A *Dedekind-infinite* set is a set whose elements can be put into a strict one-to-one correspondence—modernly termed a bijection—with the elements of a proper subset of itself. This concept rather adroitly formally summarizes the known paradoxes of infinity, helping to make infinity a mathematically approachable/tractable concept/entity. This set theoretically fundamental concept has been accepted in mathematics and philosophy for well over a century.

Nevertheless, a fundamental but heretofore overlooked theorem on bijections has recently been discovered: if the pre-image and image sets of a bijection have an element in common, then one can construct a bijection from the pre-image set with the common element removed onto the image set with the common element removed. This theorem and its proof are all but trivial, and have already been informally accepted as correct by many mathematicians. They object, however, to the implications of this theorem, that it makes it possible to construct a "paradoxical bijection" from a non-empty set onto the empty set, reminiscent of the now time-honored Banach-Tarski Paradox, that a 3-dimensional solid ball can be divided up into a small number of pieces that can be rigidly rearranged to form a solid ball of any size. This new theorem raises questions about the soundness of the foundational concept of "Dedekind-infinite", and strongly suggests that opening a communities-wide deconstruction of set theory—both philosophical and theoretical—is in order.

1. Introduction: Poincaré et al and Cantor's set theory

J. Henri Poincaré (1854–1912), one of the greatest mathematicians of all time, is often credited with having characterized the set theory founded by Georg Cantor (1845–1918) as a "grave mathematical malady, a perverse pathological illness"[1] which was infecting the discipline of mathematics and from which mathematics would eventually be cured. If in fact Poincaré did hold such a rather negative

[1] Dauben 1979, 1; a fascinating intellectual biography of Cantor. Kline 1972, Vol. 3, 1003.

opinion[2], he was not alone. Other preeminent mathematicians also condemned or held strongly negative opinions about Cantor, his set theory, and/or some of its fundamental principles. It is unfortunate for both philosophy and mathematics that most of the controversy devolved rather more on ad hominem than on "ad theorem":

Kronecker (1823–1891) publicly vilified his former pupil as a "scientific charlatan", a "renegade", and a "corrupter of youth".

Brouwer (1881–1966) said (post -Cantor) "[Cantor's theory as a whole is] a pathological incident in history of mathematics from which future generations will be horrified."

Weyl (1885–1955) said (post Cantor) "[the axiomatic set theory is a] house built on sand."

Wittgenstein (1889–1951) said (post Cantor) "Cantor's argument has no deductive content at all."

Gauss (1777–1855), however, held (pre-Cantor) the rather more philosophical than mathematical opinion that "I must protest most vehemently against [the] use of the infinite as something consummated, as this is never permitted in mathematics." The terminology later changed to "actual infinite", "completed infinity", and such, but mistrust remained among many. Gauss, unfortunately, did not give his philosophical stand any truly mathematically theoretical basis. Perhaps it was too early for that.

Cantor, however, also had eminent supporters such as Richard Dedekind, Bertrand Russell (1872–1970), and, most famously, David Hilbert (1862–1943) and his exaltation of "the paradise which Cantor has created." Cantor's set theory has Kuhnianly outlived its once openly vocal opponents, so far.

Although this all seems so far in the past as to be fruitlessly moot and passé today in the second decade of the 21st Century, a new and heretofore overlooked theorem concerning bijections has been discovered (see section 3) that bears heavily on this now seemingly atavistic controversy. The theorem is so simple and fundamental that one must question how it could have been overlooked, and what would have happened if it had been raised when Dedekind proposed his concept of an infinite set, now known as "Dedekind-infinite", or later, when Cantor proposed his simple semiformal proof of the theoretical existence of such a set within his set theory.

The intention here is to instigate and initiate the arguably well-defined process of "deconstruction" on a community-wide basis, where by "deconstruction" is here intended a process of critically—even "forensically"—reviewing and analyzing some of the fundamental assumptions, warranted or unwarranted, explicit and implicit, in the mathematics and philosophy of set theory. The aim is to present new

[2] Gray 2008, 262, expresses an interesting and very different opinion on the historical veracity of any expression of any such belief by Poincaré. The precise historical details of what Poincaré—or Gauss, or anyone else—actually did or did not say, however is not a relevant issue here. The quotes—and/or misquotes—are being used here merely to launch the "deconstruction" of set theory.

theorems, new arguments, and new analyses of these fundamentals of set theory, theorems, arguments and analyses that were somehow overlooked, not only by set theory's proponents, but also, and far less understandably, by set theory's opponents as well. Some of the immediate results of these analyses, conducted "deconstructively outside the usual box", can be considered "paradoxical" in the context of set theory's explicit fundamental tenets. These results will be controversial, as much that has been called deconstruction has been so far, but the community should find itself the better for exploring the controversies, old as well as new, as often happens. These results strongly suggest that as a community we truly need to open a full and public deconstruction of set theory, to reopen old and seemingly "completely" resolved questions about the soundness of the foundations of set theory, the concept of Dedekind-infinite in particular, to re-evaluate them thoroughly, and to perhaps choose new directions for the developments and evolutions of our current set theory and its philosophy of the infinite.

> *"When a long established system is attacked, it usually happens that the attack begins only at a single point, where the weakness of the doctrine is peculiarly evident. But criticism, when once invited, is apt to extend much further than the most daring, at first, would have wished."*
> *Bertrand Russell*[3]

> *"We have put a fence around the herd to protect it from the wolves but we do not know whether some wolves were already enclosed within the fence."*
> *J. Henri Poincaré*[4]

2. The 19th Century sees infinity enter mathematics

Infinity came to be formalized in two ways in the 19th Century. The ancient infinity that originated from counting— 1, 2, 3,...—was formalized as the Axiom of Infinity, with 0 only added later. (Peano, for example, used 1 as the "first" natural number instead of 0 in his original formulation of his axioms.) The earliest and most "intuitively obvious" (a dangerous term) variant of this axiom posited the existence of a set defined/constructed by two rules: 1 is an element of the set, and if n is an element of the set, then $n+1$ is also an element of the set. (There is an implicit third rule, that the set has no other elements.) This constructs a set, often called (or considered equivalent to) the set of all natural numbers, $\mathbb{N}\equiv\{1,2,3,...\}$, that is infinite in the most common pre-mathematical-formalization sense. In set theory, the transfinite number that represents the size or "*cardinality*" of this set $\mathbb{N}$ Cantor called "$\aleph_0$" (read "aleph-null", sometimes "aleph-naught" or "aleph-zero").

Importantly, the fact that this newly theoretically defined (as a concept) "set", theoretically defined (in its instantiations) by its members/elements, meant that this

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[3] Russell 1897, 1996, Chap. I, 17.

[4] Kline 1972, Vol. 3, 1186.

set was an “actual” or “completed” infinity of these elements. Theoretically the set would never need to have—theoretically it never could have—more elements added to it, as would need to be added to it if it were an Aristotelian “potential infinity”. This “consummated infinite”, this “completed infinity”, is what Gauss, Poincaré and many others objected to, often strongly, and, along with impredicativity, all but exclusively. (The concerns that so many had—and still have—about impredicativity, the property of a self-referencing definition, could have been dealt with much more gracefully and competently if recursion theory had been well understood at that time, and since, especially its concept of partial recursion. After all, impredicativity is not merely to be *found* in recursion theory, it is the primary fundamental essential of recursion itself. The real problems with impredicativity arrive only with *partial* recursion. Strangely, no one in the community today seems to note this publicly.)

Infinity also came to be formally defined in a second way: by its ancient paradoxes. Ancient, hallowed, sacrosanct by their great age, these paradoxes were, and still are, held to be unquestionably inherent in infinity—but of course to be questioned, forensically reexamined, in our budding deconstruction of set theory. The most commonly known such paradox is that there is the same (cardinal) number of even natural numbers as there is of both even and odd natural numbers because they can be put into the one-to-one correspondence $n \leftrightarrow 2\cdot n$ for all natural numbers n in $\mathbb{N}$. Galileo pointed out that the squares and cubes of each of the natural numbers gives more-or-less the same paradox, but he was trying to discourage people from thinking of infinity in such paradoxical terms because of the—to him—“intuitively obvious” absurdum of the reductio.

It was Dedekind who first gave the most general mathematical formalization of these ancient paradoxes. He proposed (then using terminology that even historians of mathematics are often not familiar with today) that a set was infinite—now called “Dedekind-infinite”[5]—if its elements could be put into a strict one-to-one correspondence with the elements of a proper subset of itself, as we saw just above in some particular cases of natural numbers. Such a strict one-to-one correspondence, modernly called a “bijection” from a “pre-image set” onto an “image set” (both unordered), proves that both sets have the same cardinality, or, equivalently, are “equinumerous”, “equipollent” or “equipotent”.

Cantor went further and made a theorem-proof in his set theory that the set $\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$ could be bijected onto its proper subset $\mathbb{N}=\{1,2,3,\ldots\}$, an instance of a Dedekind’s formalization of the ancient paradoxes of infinity, by mapping n in $\{0,1,2,3,\ldots\}$ one-to-one onto $n+1$ in $\{1,2,3\ldots\}$. Although it is not the usual terminology, we can call such a bijection a “*Dedekind-infinite bijection*”. This paradigmatic Dedekind-infinite bijection of Cantor’s is also taken as a proof of perhaps the most fundamental result of transfinite (cardinal) arithmetic, that

[5] Tiles 1989, 62. Wikipedia, “Dedekind-infinite set”.

$\aleph_0 + 1 = \aleph_0$.[6] Cantor wanted an infinity that could not be made larger by adding 1, thus his $\aleph_0 + 1 = \aleph_0$, and early on he even tried to define his concept of cardinal infinity in that same way, as the first—subsequently any—cardinal number that could not be made larger by adding 1. Though equivalent to Dedekind's concept, this never became popular for reason of its informality.

The strange thing that happened then, a la Sherlock Holmes, was that no one then openly suggested that these anciently venerated paradoxes might actually be an expression of ancient mathematical naiveté rather than an inherent property of infinity, a Trojan Horse in which were hiding Greek warriors ready to insidiously destroy the Troy of our then budding set theory after being ensconced within the city walls. No one openly questioned whether these ancient paradoxes of infinity could be successfully incorporated, principally through their formalization by Dedekind, into sound mathematical foundations, especially with regard to the then—and still—all-important concept of theoretical consistency. No one suggested that these paradoxes, at least as formalized by Dedekind, needed a serious vetting for theoretical validity using the other newly evolving formal tools being developed by Cantor, et al. And most importantly, no one made any public attempt at any such vetting. (Ad hominem does not really count as vetting.) We will here commence such a vetting, paradigmatically inspired by the still popular but probably overly-defined concept of "deconstruction", a vetting/deconstruction such as might have occurred if Poincaré, Kronecker, Dedekind, Cantor, Hilbert, Russell, et al had not overlooked the necessity of such a venture.

3. Opening a deconstruction of Dedekind-infinite bijections

Here the simple but overlooked fundamental theorem concerning bijections mentioned earlier is informally but clearly presented. We look at the most general case, that of arbitrary bijections (not necessarily Dedekind-infinite) where the pre-image and image sets have at least one element in common. (Note that a bijection is formally from a set onto a set, but we will informally refer to "subbijections" from elements onto elements. There should be no cause for confusion in doing so.)

We will note in passing that today's standard concept of a permutation as the ordering of the elements of a set or the alteration of such an ordering, currently formalized either as an arbitrary bijection from a set onto itself or as the transformation of one such bijection to another, can easily and fruitfully be generalized, extended, and applied to bijections in general. Simple variants of these generalizations/extensions will be used in the proof of this theorem. However, even a basic study of these is beyond the scope of this paper.

THEOREM : Given a bijection $B(SP,SI)$ from a pre-image set SP onto an image set SI, where SP and SI have at least one element EC in common, then using

[6] Cantor 1915, §6, (2), 104.

only simple bijectivity preserving operations one can construct a bijection B' from $SP-\{EC\}$ onto $SI-\{EC\}$, i.e. $B'(SP-\{EC\},SI-\{EC\})$.

PROOF : In constructing this new bijection B' we have only 2 possible cases for the common element EC:

1) If the common element EC is already paired with itself (subbijected onto itself under the bijection), then we can entirely remove this identity pairing (the identity subbijection of the pre-image EC onto its image self), and what remains will trivially be a bijection from SP-$\{EC\}$ onto SI-$\{EC\}$, the new bijection $B'(SP\text{-}\{EC\},SI\text{-}\{EC\})$.

$$\left\{\begin{array}{cccc} EP_1 & \mathbf{EC} & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ EI_1 & \mathbf{EC} & EI_3 & \dots \end{array}\right\} \Rrightarrow \left\{\begin{array}{ccc} EP_1 & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow \\ EI_1 & EI_3 & \dots \end{array}\right\}$$

Figure 1 Bijectivity preserving elimination of the EC identity subbijection

Bijectivity is trivially preserved by this operation. (The removal of a subbijection is an example of generalizing and extending the standard concept of "permutation" as the bijection of a set onto itself to more general bijectivity preserving operations on bijections.) In particular, we need not “reorder” (a la Cantor) any elements of SP or SP-$\{EC\}$ with respect to SI or SI-$\{EC\}$.

2) If EC is not identity paired with itself (subbijected onto its image self under $B(SP,SI)$), then EC in SP must be paired with some element EI in SI and some element EP in SP must be paired with EC in SI. In a (trivially) bijectivity preserving fashion, we can switch the pre-image elements EC and EP (a standard permutation, thought of as an operation, except that the pre-image and image sets are here not in general the same).

$$\left\{\begin{array}{cccc} EP_3 & EP_2 & \mathbf{EC} & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \mathbf{EC} & EI_2 & EI_3 & \dots \end{array}\right\} \Rrightarrow \left\{\begin{array}{cccc} \mathbf{EC} & EP_2 & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \mathbf{EC} & EP_2 & EI_3 & \dots \end{array}\right\}$$

Figure 2 Bijectivity preserving construction of the EC identity subbijection within the bijection by simply *switching* pre-image elements

We now have a generalized and bijectivity preserving “permutation” of B, now with the common element EC identity paired with its image self, the pre-image EP paired with (subbijected onto) the image EI, and the rest of the bijection, i.e. from SP-$\{EC,EP\}$ onto SI-$\{EC,EI\}$, remaining the same as in the original bijection. As in case 1), the identity pairing/subbijection from EC onto itself can be removed, leaving the (sub-) bijection from SP-$\{EC\}$ onto SI-$\{EC\}$, the needed bijection $B'(SP\text{-}\{EC\},SI\text{-}\{EC\})$. □

This quasi-formally described operation and its result can easily be translated into a formally rigorous theorem. This operation cannot “generalizedly permute” a valid bijection into a non-bijection. We should also note the corollary, that if one derives/-obtains a non-bijection as a result of applying this operation—or any sequence of such operations—to an ostensibly valid bijective mapping, that initial mapping cannot have been a valid bijection. This at first glance innocuously correct theorem and proof, both simple and obvious once pointed out, has never been published in any work pertaining to set theory, especially neither by Dedekind nor Cantor nor, far less understandably, by Poincaré, the greatest mathematician of that time.

Instead of railing against set theory's fundamental concept of a "consummated/-completed infinity", Poincaré et al would have done much better to initially focus their efforts on the fundamental concept/property of Dedekind-infinite that formalized the ancient paradoxes of infinity and brought them within the reach of mathematical theory. As indicated by this theorem, they would have done better to focus their initial attentions, not on Cantor's use of "consummated infinities", but on the bijections from a set onto a proper subset of itself that demonstrate that the set is thus Dedekind-infinite, the Russellian "single point, where the weakness of the doctrine is peculiarly evident".

4. Further deconstruction of Dedekind-infinite bijections

The new yet fundamental theorem just presented strongly brings into question the consistency/soundness of the theoretical *existence* of a Dedekind-infinite set, suggesting that we closely examine Cantor's proof of such an existence for possible flaws. Thus, we continue our deconstruction outside the usual box by looking carefully at his construction of his paradigmatic Dedekind-infinite bijection. We will refer to the Axiom of Infinity and its definitional sequential construction of the completed infinite set $\mathbb{N}\equiv\{1,2,3,...\}$ and the corresponding sequential construction of the bijection that proves that $\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$ is Dedekind-infinite (easily modified to show that $\mathbb{N}$ is Dedekind-infinite, as well), and contrast that result with Cantor's now standard construction of the bijection by "simultaneously" bijectively mapping n in $\{0,1,2,3,\ldots\}$ onto $n+1$ in $\{1,2,3\ldots\}$, the bijection that gives rise to that most fundamental equation/theorem in transfinite (cardinal) arithmetic, $\aleph_0+1=\aleph_0$.

The following is intended to simplify the formal arguments. For Cantor, cardinality was an abstraction with regard to any properties the elements of the set might have (order, numerical value, etc) except the property that all the elements are distinct from each other. So, for example, if we wished to turn the set $\{1,2,3\}$ of cardinality 3 into what we can call here a "*cardinalized set*" (of the same cardinality), it could be represented as, for example, "{●,●,●}" or "{▼,▼,▼}". When constructing the bijection of a pre-image set onto an image set to show that they are equinumerous, it does not matter which element in the pre-image set is "*subbijected*" onto a corresponding element in the image set. Only the property of the distinctness—and number—of the elements of the sets matters.

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Figure 3 Trying to biject 2 ▼s onto 1 ▲ by switching the unsubbijected ▼ with the subbijected ▼

In Figure 3 we see the beginning of an attempt to construct a bijection from a cardinalized pre-image set {▼,▼} of cardinality 2 onto a cardinalized image set {▲} of cardinality 1.

It is trivially obvious in Figure 3 that it is not possible to construct a bijection from {▼,▼} onto {▲}. The deconstructively interesting insight, however, has to do with the details of why this is not possible, an essential part of the larger deconstruction of Cantor's proof that the set {0,1,2,3,…} can be bijected onto its proper subset {1,2,3,…}, and is thus Dedekind-infinite. It is not possible because the image ▲ is already paired in a subbijection with a pre-image ▼, and the only legitimate way to obtain a free image ▲ to subbiject the unsubbijected pre-image ▼ onto is to desubbiject the subbijected pre-image ▼, giving us 2 un-subbijected pre-image ▼s, only 1 of which can then be (re-) subbijected onto the image ▲. In fact, we immediately notice that switching the 2 ▼s is abstractly equivalent to not switching them, or to switching them any number of times, even Cantor's "absolute infinite" number of times. (It is good to remember an essential consideration, that "abstract equivalence" is always with regard to a purpose/set of purposes, here of constructing a desired bijection.) Another way to think of this situation is that we cannot make any progress toward constructing the desired bijection by switching the pre-image elements, here one unsubbijected and the other subbijected.

In Figure 4 it is again trivially obvious that it is not possible to construct a bijection from the cardinalized pre-image set {▼,▼,▼} onto the cardinalized image set {▲,▲}, and we continue to deconstructively examine in detail why it is not possible. The newly added subbijection, {▼⇒▲} (simplifying the notation from {▼}⇒{▲}), with regard to our immediate purpose here of constructing a bijection from {▼,▼,▼} onto {▲,▲}, is abstractly equivalent to the earlier subbijection, {▼⇒▲} from Figure 3, here the left such subbijection in Figure 4. This means that we cannot make any progress toward constructing the desired bijection by switching the unsubbijected pre-image element with the subbijected pre-image element in this new subbijection, either.

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Figure 4 Trying to biject 3 ▼s onto 2 ▲s by switching (each) unsubbijected ▼ with a subbijected ▼

Switching already subbijected pre-image elements also obviously does not help us here either, so we can simplify the argument to any switching of pre-image elements, independent of whether they are subbijected or unsubbijected. And this means not only that we cannot make any progress toward constructing the desired bijection by switching pre-image elements any number of times, but that we also cannot make any progress toward constructing the desired bijection by adding any number of new subbijections {▼⇒▲}, each and every one of which will be abstractly equivalent to the first such subbijection. It should be obvious that adding an unsubbijected image element, or rather an incomplete subbijection {free/available⇒▲}, is not a valid operation with regard to our purposes here.

It is a fundamental requirement in mathematics, when defining and using defined entities, to be able to re-substitute the original definition for the defined entity and to then derive/obtain precisely the same result obtained in the derivation performed using the defined entity. In our case here we have already started to perform this re-substitution with regard to constructing a bijection from $\{0,1,2,3,...\}$ onto $\{1,2,3,...\}$. We implicitly performed this substitution of the original sequential definition of infinity given earlier—defining/constructing the infinite set $\mathbb{N}\equiv\{1,2,3,...\}$ by starting with $n = 1$ and continuing to add 1 to each new element, i.e. n in $\mathbb{N}$ implies $n + 1$ is also in $\mathbb{N}$—for the entity that this defined, "infinity", which after it is defined seems to exist "simultaneously". That is, we looked at a cardinalized view of constructing a bijection from $\{0,1\}$ onto $\{1\}$ (the $n = 1$ starting point in the definition of the infinite set $\mathbb{N}$; see Figure 3), i.e. from the cardinalized set {▼,▼} onto the cardinalized set {▲}, and then at a cardinalized view of constructing a bijection from $\{0,1,2\}$ onto $\{1,2\}$ (i.e. $n = 2$; see Figure 4), i.e. from {▼,▼,▼} onto {▲,▲}.

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Figure 5 Trying to biject $\aleph_0+1$ ▼s onto $\aleph_0$ ▼s by switching (each) unsubbijected ▼ with a subbijected ▼

As in Figure 5, if we have an infinite number of additional subbijections {▼⇒▲}, we could set up an infinite "borrow from Peter to pay Paul" situation, a limiting case pyramid scheme, but the question arises of whether we could "complete" the infinity of that process, as we would theoretically need to do. This is an essential question since the infinities and infinite sets of set theory are *all* considered to be "completed", the sets never having further elements added (as opposed to Aristotle's concept of a "potential infinity"). And to "complete" the ostensibly bijective mapping, we would need to provide our unsubbijected pre-image element with a bijective submapping without permanently depriving any subbijected pre-image element of such. But this would mean that we would catch ourselves cheating, since it implies the existence of a non-trivially unsubbijected image element, which "legally" we can neither have to start out with nor obtain later.

If we proceed in this sequential fashion in an attempt to construct an (arbitrary) bijection from $\{0,1,2,3,...\}$ onto $\{1,2,3,...\}$, i.e. from the cardinalized $\{▼,▼,▼,▼,...\}$ onto the cardinalized $\{▼,▼,▼,...\}$, as in Figure 5, it is clear that we will not obtain such a bijection "at infinity", not "at *any* infinity". Since the cardinalization of the sets merely made this process less confusing, it is clear that we cannot, if we proceed in this same fashion, construct a (valid) bijection from the "*uncardinalized*" set $\{0,1,2,3,...\}$ onto the uncardinalized set $\{1,2,3,...\}$, and for the same reason: each image element onto which we might subbiject the unsubbijected pre-image element is already bijectively mapped in a subbijection, and the only way to then subbiject the unsubbijected pre-image element onto that image element is to desubbiject the pre-image element subbijected onto that image element and subbiject the unsubbijected pre-image element onto that now *temporarily* free/available unsubbijected image element. But this process will always leave an unsubbijected pre-image element, whether or not the infinity of the "bijection" is eventually an anti-Aristotelian and anti-Gaussian "consummated infinity", as in Cantor's set theory it theoretically must be. We could clearly make this a formal theorem of set theory. (We are not concerned here with any particular axiomatization of set theory, just the fundamental concepts.)

On the other hand, as mentioned earlier, Cantor's proof is still held to show that we, in theoretical fact, can and do obtain a seemingly valid bijection from $\{0,1,2,3,...\}$ onto $\{1,2,3,...\}$ by the "simultaneous" variant of the above sequential process. Just as, after it has been defined, "infinity" seems to have a "simultaneous existence", in our usual practice we tend to think of the mapping from n in $\{0,1,2,3,...\}$ onto $n+1$ in $\{1,2,3,...\}$ as being constructed "simultaneously", which makes it seem "intuitively obvious" that the bijection has been constructed successfully. For example, we never think to look for an unsubbijected pre-image element "at infinity".

The reaction to the above of at least one member of the mathematical community was theoretically peculiar. Incredible as it may seem, a formal referee (of a well-respected journal) of a paper with a minor variant of the above analysis held that when an infinite entity, such as $\mathbb{N}=\{1,2,3,\ldots\}$ or a Dedekind-infinite bijection from $\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$ onto $\mathbb{N}$, has been defined, such a defined entity then has a "simultaneous existence"; he further held that it then becomes formally invalid to analyze such a "simultaneous" entity using sequential methods such as those that were used to define such an entity in the first place, and such as were used above to analyze/deconstruct the construction of such an entity (in that paper a Dedekind-infinite bijection). (It is from his usage in his comments that I obtained this less than standard terminology of "simultaneous" theoretical existence.)

So, our deconstruction of Cantor's proof has dis-covered a serious disparity: if we allow a Dedekind-infinite bijection to be defined/constructed "simultaneously" in such a way that it has a "simultaneous existence" that formally can no longer be validly analyzed sequentially, we get one result. But if we allow a sequential analysis, as when we re-substitute the original sequential definition of "infinity" in place of the "simultaneous" or "simultaneously existing" defined entity of "infinity", we clearly get a disparate result; i.e. we clearly find that we cannot thus successfully construct a bijection from $\{0,1,2,3,...\}$ onto $\{1,2,3,...\}$ by proceeding sequentially in accordance with the formal definition of the Axiom of Infinity. This strongly suggests that we will eventually agree that a subtle variant of circular reasoning exists in the "simultaneous" approach used by Cantor, hidden by an inherent partial recursion in same.

Further, and most importantly, this partial and informal sequential analysis/-deconstruction of the paradigmatic construction of a Dedekind-infinite bijection can easily be translated into a formally rigorous analysis resulting in the same "paradoxical" disparity as in the results we saw above. This easily discovered disparity—between the results of "simultaneously" constructing the bijection, as Cantor did, and constructing it sequentially, as we theoretically need to be able to do without obtaining a disparate result—is an essential finding in a deconstructively careful vetting of set theory's adoption of infinity and its paradoxes, especially as formalized by Dedekind, an essential disparity that was completely overlooked by Cantor, et al. This analysis has never before been published in any mathematical or philosophical work on set theory, especially in this case neither by Cantor nor, far less understandably, by Poincaré, either of whom should have found it easily.

5. Closing the deconstruction of Dedekind-infinite bijections

The above general result for bijectivity-preservingly eliminating a single arbitrary common element from the pre-image and image sets of an otherwise arbitrary bijection does not of and by itself constitute or demonstrate a "discrepancy" within set theory. The discrepancy deconstructively demonstrates itself when we apply this simple result to a Dedekind-infinite bijection from a (transfinite) set onto a (likewise transfinite) proper subset of itself. Together with a brief overview of possible concomitants, below, examining this application will constitute the closing stage of our deconstruction of Dedekind-infinite bijections.

We continue this closing stage by here exploring the psychology of applying part of the above described operation in a relatively innocuous situation, that of permutations of a set. Permutations used to be defined as different orderings of an ordered set. E.g. $\{2,1,3\}$ and $\{3,1,2)$ would be non-identity permutations of the ordered set $\{1,2,3\}$ (also its own identity permutation). Standard permutations of a set, ordered or unordered, are today defined as bijections from that set onto itself.

Given an arbitrary bijection from an arbitrary non-empty set, S (of any cardinality), onto itself, i.e. an arbitrary standard permutation of S, if we apply merely the second part of the operation given earlier, that of switching pre-image elements so as to identity re-pair/re-subbiject at least one pre-image element with each such switch (seen in Figure 2), then using a possibly transfinite succession of such switches, at most one for each and every element of S (all of which are common for standard permutations), we can re-construct the identity bijection (identity permutation) starting from *any* arbitrary bijection from the given set, S, onto itself. The first level argument for this seemingly quite obvious result is that if we could not identity re-pair each and every element (all common) in the pre-image and image sets using this operation, there would have to exist at least one (common) element that we could not thus identity re-pair using this operation. But this would contradict the already proven generality of the operation and the result of its application, proving by contradiction—here only semi-formally—that all the common elements can thus be identity re-paired.

We can also fortify this argument using mathematical induction (finite or transfinite). If we have a bijection from $\mathbb{N} \equiv \{1,2,3,...\}$ onto itself, i.e. a standard permutation of $\mathbb{N}$, of transfinite cardinality $\aleph_0$, we can use finite induction[7] to lend further authority to the identity re-pairing/resubbijecting, first of 1, then of 2, then of 3, and so on through the entire completed infinity, $\aleph_0$, of natural numbers in $\mathbb{N}$, yielding the identity permutation of $\mathbb{N}$. If we removed each identity pairing/identity subbijection, e.g. as it was formed, we would wind up with the limiting case of a trivial standard identity permutation of the empty set (its bijection onto itself).

The deconstructively critical thing to note here is that this overall operation of restoring the identity (sub-) bijection of the natural numbers onto the natural numbers does not in fact depend on the *pre*-image set being restricted to the set of all natural numbers $\{1,2,3,...\}$ (and the bijection being restricted to a standard permutation). The pre-image set could be, for example, the set $\{0,1,2,3,...\}$ even though the image set continued to be $\{1,2,3,...\}$, and the above argument would remain valid.

An important thing to note about switching pre-image elements of an arbitrary standard permutation so as to identity re-pair/identity resubbiject all the (perforce common) elements is that we *cannot*, proceeding in this way, derive or otherwise obtain a contradiction that would demonstrate theoretical inconsistency (in the case of a standardly defined permutation). In addition, neither the switching of pre-image elements nor the removing of identity subbijections (not a standard operation in regard to standard permutations) can "generally permute" a valid permutation (or a valid bijection) into an invalid permutation (or a non-bijection). So there is nothing about performing such an identity re-pairing or identity resubbijecting an infinite number of times that can *introduce* an inconsistency into a theory, although it might help us discover such, just as a valid audit might discover embezzlement without ever being able to cause, create or otherwise engender an embezzlement.

The salient point here has to do with the psychology that starts to emerge at this point. It is disconcerting to many that, if all the common elements are removed from

[7] Borowski and Borwein 1991, 222.

a Dedekind-infinite bijection, using a bijectivity preserving operation that cannot construct an invalid bijection from a valid bijection, there remains a "paradoxical bijection" from a non-empty set onto the empty set. I.e. this extended sequential operation constructs a "paradoxical bijection" that is strongly reminiscent of the Banach-Tarski Paradox[8]. There is even a very strong hint of such a "paradoxical bijection" being theoretically related to it, since from $\aleph_0 + 1 = \aleph_0$ one formally derives $n \cdot \aleph_0 = m \cdot \aleph_0$ and $n \cdot 2^{\aleph_0} = m \cdot 2^{\aleph_0}$, strikingly similar to the Banach-Tarski Paradox that a ball the size of a pea can be cut into a small number of pieces[9] (not "geometrically intuitive" to non-mathematicians, or even to many mathematicians) which can then be rearranged to form 2 solid balls of the same size, or a solid ball of any size, even the size of the sun.[10] By the way, the Banach-Tarski Paradox is still generally accepted as a legitimate paradox of infinity in set theory (set theoretic geometry) instead of a contradiction proving theoretical inconsistency.

6. Formal consequences of the formal definition of a mathematical theory

The most common psychological reaction among mathematicians to the derivation of such a "paradoxical bijection" has so far been that such a "paradoxical bijection" is a contradiction, and as such it proves that the *derivation* of this selfsame contradiction must be invalid. This reaction can be considered unfortunate because any mathematical theory, by definition, consists of (ignoring formal languages) the axioms, the rules of inference, and *all* the theorems that can *possibly* be derived from the axioms using the rules of inference. And a mathematical theory is formally inconsistent if a contradiction, such as that demonstrated above, can even *possibly* be derived within the theory.

This means that, if a mathematical theory is inconsistent, then it is theoretically possible to *validly* derive a contradiction within the theory. Thus, an essential consequence of the formal definition of a mathematical theory is that, formally theoretically, a contradiction can never—of and by itself—be taken to demonstrate that the derivation of that selfsame contradiction is invalid. A contradiction derived completely within the theory (using only the basic assumptions of the theory, usually just the axioms and rules of inference) always proves that the theory is inconsistent. The derivation that *seems* to *generate* the contradiction really merely *demonstrates* the contradiction, and the inconsistency of the theory. We must never allow ourselves to put mathematics in a regime where the "auditor" and/or "auditing process" is blamed for the "embezzlement". (By the way, this is much the same situation as with the Axiom of Choice, which allows a gelded form of "auditing", which "auditing process" Tarski blamed for the "embezzlement"—obvious to himself—in the Banach-Tarski Paradox.)

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[8] Wagon 1993.

[9] It was shown in the 1940s that 5 pieces suffice, but no fewer.

[10] Wapner 2005.

But set theory was evolved in essential part out of the ancient paradoxes of infinity, by way of the newly formalized concept of "sets", and theoretically incorporated these paradoxes without any noteworthy attempt at resolving them. This made it—and still makes it—psychologically easy to classify any derived contradictions as demonstrations of the inescapable paradoxes inherent in infinity, and thus to take any contradiction itself as a proof of the theoretical invalidity of the derivation of that selfsame contradiction, or at least of its result, except insofar as it is considered a valid instance of such an inherent paradox of infinity, a paradigmatic example of which is the Banach-Tarski Paradox.

Psychological reactions to the above derivation of "paradoxical bijections" have gone even further, even to the point of abandoning mathematical induction. (Of course, if one abandons mathematical induction, one likewise abandons both the Axiom of Infinity and set theory. See below.) If the Dedekind-infinite bijection is, for example, from the pre-image set $\mathbb{N}\cup\{0\}=\{0,1,2,3,\ldots\}$ onto the image set $\mathbb{N}=\{1,2,3,\ldots\}$, then we can use the finite induction to demonstrate that all the natural numbers in $\mathbb{N}$ can be identity paired and removed using the bijectivity preserving operation(s) given above, leaving a "paradoxical bijection" from the non-empty set $\{0\}$ onto the empty set. This argument is even more cogent than the most basic one given earlier when applied to the special case of the set of all the common elements being $\mathbb{N}$. But the reaction of some members of the mathematical community to this is theoretically peculiar: some mathematicians (formal referees of well-respected mathematics journals) responded that, in general, when finite induction is used and a contradiction (such as this one) is derived, then finite induction is *formally theoretically limited* to proving the result only for a "finite (proper) subset of the natural numbers". One mathematician (the editor-in-chief of a respected mathematics journal) even responded that finite induction *always* only proves the result for a "finite (proper) subset of the natural numbers", *never* for *all* of them.

Finite induction[11], it should be remembered, is the conjoined identical twin of the Axiom of Infinity: in the Axiom of Infinity the predicate that is constructed/"proven" for each and every natural number is its membership in the set of all natural numbers; in finite induction, the predicate that is to be proven is chosen at the time of the finite induction based proof. (Since the above bijectivity preserving common element elimination is so general, one could also apply it using transfinite induction to give the most general result.) Standard finite induction—when starting with 1 and successfully applied—always proves the predicate for each and every natural number in the transfinite set of all natural numbers[12], never for merely a "finite (proper) subset" of the natural numbers. If one starts to partially or wholly abandon finite induction, one implicitly but equally starts to partially or wholly abandon both

[11] Borowski and Borwein 1991, 222.

[12] Borowski and Borwein 1991, 222.

the Axiom of Infinity and set theory. The term “deconstruction”, nevertheless, seems to pale in comparison with the above responses of respected community members.

Here we close the deconstruction of Dedekind-infinite bijections, noting that if the above deconstructive analyses had been made public by Poincaré (or Kronecker, or anyone with significant credentials), set theory would never have gained community acceptance as a mathematical theory. And we move on, first to a quick deconstructive look at standard proof by contradiction, and then to initiating/-exhorting the opening of a general, community-wide deconstruction of set theory.

7. A quick, partial deconstruction of proof by contradiction

Proof by contradiction, also known as indirect proof or apagogical argument, is a non-equivalent variant of the ancient “reductio ad absurdum”. One starts with a theory and an intended theorem to prove in it (ostensibly not already known to be a theorem of the theory), then one takes the negation of the intended theorem (together with the whole of the theory) and derives a contradiction that is considered to indirectly prove the intended theorem in that theory.

We will quickly note that proof by contradiction does not take into account Gödel’s completeness and consistency results. Anciently, arguments assumed both completeness and consistency. Ever since Gödel’s results, *incompleteness* is known or assumed for most actual mathematical theories, but with the consistency of the theory also assumed, despite the fact that, per Gödel, consistency can never be effectively proven, although it may eventually be falsified, a la Karl Popper. It helps to remember that after Gödel had proven that the Continuum Hypothesis was independent of Zermelo-Fraenkel, Cohen found it unwise to passively accept that Gödel’s result implied that the negation of CH must also therefore be independent of ZF; he carefully proved that further result also.[13] After Gödel’s—and Cohen’s—results, proof by contradiction should have come up for immediate review and reevaluation, but that has yet to happen.

Proof by contradiction has a further characteristic that concerns us here. It must assume the consistency of the theory that the theorem is to be proved in. If we are dealing with a theory when its consistency is in serious doubt, we must philosophically and theoretically mistrust proof by contradiction. In any theory that is inconsistent it is theoretically possible to produce a contradiction, and to do so even before we start to attempt a proof by contradiction. I.e. if we attempt to prove A by contradiction, we must be able—theoretically—to find a contradiction in the theory before any logical/theoretical inclusion of either A or the negation of A in the argument or derivation takes place (unless A or its negation is already a theorem and is explicitly or implicitly used). Thus from yet another logical/theoretical direction in addition to the usual, when a theory is inconsistent it is theoretically possible to prove any possible theorem simply by using proof by contradiction.

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[13] Cohen 1963, Chap. 4.

And this again tells us that we may never take the derivation of a contradiction to per se prove that that selfsame derivation of the contradiction must be invalid. We must never, upon deriving a contradiction from within the theory, then deny that it in fact was derived from within the theory, i.e. we must never use a pseudo proof by contradiction to argue "assume that we can validly derive the contradiction just obtained from within the theory; since we obtain(ed) a contradiction, we therefore cannot validly derive that contradiction just obtained from within the theory." This is the traditional schoolboy proof that all mathematical theories are provably consistent—Gödel to the contrary notwithstanding—because, as just shown, any time one derives any contradiction from within the theory, it can then be "proven by contradiction" that it could not have been "validly" derived from within the theory by assuming that it could be validly derived from within the theory and then using that selfsame contradiction to invalidate that assumption, thus proving that no contradiction can ever be validly derived from within the theory.

In fact, if there is any suspicion of inconsistency, we would be advised to start fresh from the axioms and rules of inference of the theory and re-derive everything without any use of proof by contradiction. All too often, once a theorem is "proven", the psychology is to reject any contradictory theorem as "necessarily" having been derived "invalidly", unless it has some "special property", to which is attached a sort of agnatic primogeniture. Agnatic primogeniture has no theoretically valid place in a mathematical theory.

8. Toward opening a general, community-wide deconstruction of set theory

The above deconstruction of Dedekind-infinite bijections rather suggests that their theoretical existence in set theory is problematic. The possible theoretical consequences are distressing, and the above results more than suggest that it will be good to open up a general, community-wide deconstructive forensic examination of set theory, to reevaluate its mathematical soundness.

It is possible that we are looking at a "falsification"—a la Karl Popper, as foreseen by Poincaré, Brouwer, et al—of set theory. This perhaps explains the psychological, as evidenced by the "theoretical", reactions of formal referees and editors to the above deconstruction of Dedekind-infinite bijections. We will look here at some of the possible theoretical consequences.

If the community eventually finds the above "paradoxical bijections" to be as standardly derivable within set theory as the Banach-Tarski Paradox is held to be, as seems reasonable from the above that they should, set theory will not just want, but need, to evolve in rather drastic ways. One of the most likely ways is that we would need to acknowledge that any set can be made larger "cardinally" by adding a new element, and our concept of "cardinal" would need to be revised, together with our concept of "ordinal". Cantor's hope for an infinity that could not be made larger by adding 1 would be dashed, and the paradoxical "Grand Hotel" that Hilbert constructed in that famous "paradise" would have to be "deconstructed".

Since the Axiom of Infinity, and the concept of a "completed/consummated infinity" that it embodies, depend on the set that it defines/constructs being Dedekind-infinite, and since that concept, at first healthy deconstructive glance, seems to succumb to same, the Axiom of Infinity likewise seems to be in line to succumb, and the concept of a "completed/consummated infinity" along with it. If we try to maintain our desire for a "completed infinity" we are all too likely to wind up with it necessarily being a dissatisfying arbitrarily chosen "infinity" that can be fuzzily approached, arrived at, and surpassed. Or perhaps it would be something like Cantor's "absolute infinite", which even Cantor considered to be an inherently inconsistent theoretical entity, albeit he embedded it in his set theory. We will also feel compelled to reconsider Aristotle's potential infinite when contemplating new set theories. We can see here that any 19th Century (or later) mathematician who was dissatisfied with the concept of a "consummated/completed infinity" should have first detoured to "deconstruct" or otherwise closely and "forensically" analyze the concept of a "Dedekind-infinite set".

Given the above possibilities, it seems anticlimactic to mention that the Continuum Hypothesis is likewise in line to be falsified, since any set seems to be made "cardinally" larger by adding a new element, i.e. its cardinality seems to always be made by larger by adding 1, it follows that $\aleph_0 + 1 > \aleph_0$, falsifying CH. And CH is likely to be thus falsified in a strong way since the "cardinality of the continuum" starts to be seen as a contradiction in terms once the concept of Dedekind-infinite sets succumbs to deconstruction, and any set can be shown to be made cardinally larger by adding a new element. It is difficult to foresee a new revised set theory in which CH, or any reasonable variation on that theme, could be successfully resurrected. One of the key changes to philosophy and theory implied by this situation would be to realize that a "continuum" can never be made to "consist" of points or sets of points, since the cardinality of the set of points between any 2 distinct points can always be made larger by there adding new points, independent of their distribution.

This leads us to the desirability of starting to evolve the concept of a "quantinuum" that can be "embedded" in a "continuum" (which needs to be completely re-conceived), or rather of many "quantinua" (suggested by infinite base expansion reals where the bases are "incommensurate" due to their prime number decompositions, e.g. base $10 = 2 \cdot 5$ versus base 3) and possibly "continua", since we might have more than one quantinuum to embed and more than one continuum in which to embed them, giving us vast and vastly interesting topological possibilities. The deconstruction of Dedekind-infinite bijections given earlier also impacts real number theory beyond CH, since our concept of infinite decimal expansion real numbers with $\aleph_0$ decimal places to the right of the decimal point falters.[14]

[14] Knowles 2004.

We can also note here that a "complete" deconstruction of set theory will also necessitate a reevaluation of the Axiom of Choice. AC really just allows a limited form of "auditing", which Tarski blamed for the "embezzlement" that he found taking place in his/their Banach-Tarski Paradox. Further, many have overlooked a consequence of the independence of the Axiom of Choice from the Zermelo-Fraenkel axiomatization of Cantor's set theory. We all seem to assume that, just as the power set of a set always exists (although this is by axiom), we all assume that the Cartesian product or product set of an infinite family of non-empty sets "exists" in the sense that, not only is the product set non-empty, it has "all" the elements (e.g. points) that it "should" have. A common example is an infinite dimensional Euclidean space (independent of any questions of metrics), which we customarily assume has "all" its points. But if this product set exists, it is a single set, and thus, without needing AC, we can choose a single "arbitrary" element/point in that set, one that is equivalent to a non-empty choice set made up of an "arbitrary choice" from each set in the infinite family of non-empty sets that went into making up the Cartesian product set. Thus, we either have that the theoretical existence of non-empty Cartesian product sets of infinite families of non-empty sets allows us to prove AC within ZF, i.e. to prove that AC is not independent of ZF, *or* we have that the independence of the Axiom of Choice from ZF proves that Cartesian product sets of infinite families of non-empty sets must themselves be *completely* empty, since even a single guaranteed element in such product sets is enough for one to prove the Axiom of Choice in ZF. This is a further indication of the need for a community-wide deconstruction of set theory.

Dauben's intellectual biography of Cantor[15] often refers to Cantor being a religious mystic who sought God, not in, but ever beyond the infinite succession of ever greater infinities produced from the power set operation, all the way to the China of Cantor's "absolute infinite", which Cantor held to be inherently inconsistent, just as God—for Cantor—was both inherently infinite beyond all possible infinities and inherently inconsistent beyond all humanly conceivable consistencies.[16] Cantor probably would have done better to try to find God in the parable of finding the lost sheep, even if it was lost in an "absolute infinity" of sheep and wilderness.

It will take a while, and much careful deconstruction and "forensic" analysis, before the community will be able to decide whether Poincaré, Brouwer, Gauss, et al were completely right—or even sufficiently right—about Georg Cantor's set theory. But there do seem to be sufficient grounds for undertaking a community-wide deconstruction of set theory. In any case, this first round deconstruction primarily of

[15] Dauben 1979, references scattered throughout; see index.

[16] Dauben 1979.

set theory's Dedekind-infinite sets and related bijections suggests important new directions for the evolution of future set theories, and the need for them.

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