Does the Banach-Tarski Paradox have an Evil Twin?!

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Abstract

New theorem on bijections: if the pre-image and image sets SP and SI of a bijection $B(SP,SI)$ have an element EC in common, then one can construct a bijection from the pre-image set with EC removed onto the image set with EC removed, i.e. $B^*(SP\text{-}\{EC\}, SI\text{-}\{EC\})$. Simple proof: if EC is identity subbijected onto EC under B, this identity subbijection is removed, trivially constructing the desired bijection, $B^*(SP\text{-}\{EC\},SI\text{-}\{EC\})$. But, if EC is subbijected onto some other image element EI, then some other pre-image element EP is subbijected onto EC. We switch the pre-image EC and EP, preserving bijectivity. This yields a bijection $B'(SP,SI)$, with EP subbijected onto EI, and EC identity subbijected onto EC. Again this identity subbijection from EC onto EC is removed from $B'(SP,SI)$, trivially constructing the desired $B^*(SP\text{-}\{EC\},SI\text{-}\{EC\})$. We apply this theorem to a "Dedekind-infinite bijection" (a bijection from a set SD onto a proper subset of itself, showing that SD is Dedekind-infinite) so as to remove all common elements. We obtain a "Paradoxical Bijection" from a non-empty set onto the empty set, the Evil Twin of the Banach-Tarski Paradox, and a challenging new paradox for the community.

bijection, bijection permutation, Dedekind-infinite, Dedekind-infinite bijection, Banach-Tarski Paradox, paradoxical bijection

MSC [2000] Primary 03E99; Secondary 03A05, 00A30

Audience

This paper and its presentation (1079-03-89) at the 2012 Spring Southeastern Sectional Meeting (#1079) of the American Mathematical Society at the University of South Florida in Tampa, FL, on Saturday, March 10, are intended for a general audience familiar with the basics of transfinite set theory. It will help to know in advance what the Banach-Tarski Paradox is, what Dedekind-infinite sets are, what bijections are, and that they are essential to set theory.

Introduction

In 1924, Stefan Banach (1892-1945) and Alfred Tarski (1901-1983) astonished the world with their Banach-Tarski Paradox[1]. In this paradox (see Figure 0), a solid 3-dimensional ball is decomposed into a finite number of (geometrically non-intuitive) pieces that can be rearranged into 2 balls the same size as the first ball. A number of interesting variants have been presented since then. Tarski blamed the Axiom of Choice, as many have over the years, for this paradox, and much else. Almost everyone else has happily accepted the Banach-Tarski Paradox as one of the most interesting paradoxes inherent in infinity.

Figure 0: The Banach-Tarski Paradox: 1 solid ball becomes 2 solid balls the same size (Sans permission from http://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox.)

As some authors note, the Banach-Tarski Paradox implicitly depends on[2], besides being similar to, the concept of *Dedekind-infinite*. Two examples of this important similarity are that the natural numbers aka the positive integers can be doubled to get all the positive and negative integers, and the unit interval of real numbers [0,1] can be doubled to get [-1,1] (each interval having the same number/cardinality of real numbers). In the late 1800s, around the time Georg Cantor (1845–1918) was getting ready to give birth to his own transfinite set theory[3], Richard Dedekind (1831-1916) proposed not only the concept of "strict one-to-one correspondences" (what we modernly call bijections from pre-image sets onto image sets) from infinite sets onto (necessarily also) infinite subsets, which strict one-to-one correspondences demonstrated that they were equinumerous (for which Dedekind used the term "similar"), he also proposed that this property of similarity/equinumerosity of a set with a proper subset of itself was not merely characteristic, but *definitional* of infinity. Sets that have this theoretical property are now known as *Dedekind-infinite* sets. We will coin a new term and call a bijection from an infinite set onto a proper subset of itself, i.e. one that demonstrates that the set is Dedekind-infinite, a "*Dedekind-infinite bijection*".

Dedekind's concept did in fact adroitly formalize mathematically the known paradoxes of infinity (some quite anciently known), and did so in the more general terms of the newly evolving set theory rather than being restricted to natural numbers, as had been previously been the case. This greatly aided the entrance into mathematics of infinity, as defined by its paradoxes—and made inevitable the entrance of those very same paradoxes. Although infinity was and still is axiomatically defined by what is now known as the Axiom of Infinity,

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[1] Wagon 1993. Wikipedia, "Banach-Tarski Paradox"

[2] Su 1990, p 3.

[3] Cantor 1874. Cantor 1915.

Dedekind's concept became an essential cornerstone of the incorporation of infinity into the foundations of mathematics, primarily in the set theory of Georg Cantor.

Many worried greatly about this acceptance, since amusing and engaging paradox can be extremely difficult to distinguish from excruciating and unendurable inconsistency. But most did decide (later) that they *could* distinguish the paradoxes inherent in infinity from contradictions that would have made mathematics inconsistent, still a theoretical calamity (despite recent developments of concepts of paraconsistency). After all, these paradoxes were so inherent in infinity that infinity itself could be mathematically defined in terms of them. Thus, with loud protest but without serious contest, infinity *and* its paradoxes were formally adopted by mathematics and welcomed into Cantor's new set theory with open arms by some, by many in fact. Cantor had eminent supporters, including Dedekind of course, but also Bertrand Russell (1872–1970), and, most famously, David Hilbert (1862–1943) and his quasi-religious exaltation of "the paradise which Cantor has created." Cantor's set theory has Kuhnianly[4] outlived its once openly vocal opponents, so far.

But Dedekind, Cantor, Hilbert, Russell, and many others, overlooked an elementary theorem that bears on Dedekind's concept. When applied to Dedekind-infinite bijections, this theorem leads to "Paradoxical Bijections", a paradox so similar to and so intertwined with Banach-Tarski that it will eventually become known as "Banach-Tarski's Evil Twin".

An example of a simple bijectivity preserving operation on bijections

If we have a simple bijection from the set $\{A,B,C\}$ onto the set $\{E,D,C\}$ (it helps to think of them as ordered), such as in Figure 1, we see that is trivial to remove the common element subbijection $C \Rightarrow C$, in a completely bijectivity preserving fashion, to trivially construct a bijection from $\{A,B\}$ onto $\{E,D\}$, i.e. from the original pre-image set $\{A,B,C\}$ with the common element C removed onto the original image set $\{E,D,C\}$ with the common element C removed.

$$\left\{\begin{matrix} A & B & \mathbf{C} \\ \Downarrow & \Downarrow & \Downarrow \\ E & D & \mathbf{C} \end{matrix}\right\} \Rrightarrow \left\{\begin{matrix} A & B \\ \Downarrow & \Downarrow \\ E & D \end{matrix}\right\}$$

Figure 1 Bijectivity preserving elimination of the $\boldsymbol{C \Rightarrow C}$ identity subbijection

Likewise, if we have a simple bijection from $\{C,B,A\}$ onto $\{E,D,C\}$ as in Figure 2, we see that is trivial to switch the pre-image elements C and A, in a completely bijectivity preserving fashion, giving us again the $C \Rightarrow C$ subbijection we have on the left in Figure 1. As before we can remove this newly formed common element subbijection $C \Rightarrow C$ to again trivially construct a bijection from $\{A,B\}$ onto $\{E,D\}$. It is easy to see that bijectivity is trivially preserved by this overall operation and each of its sub-operations.

[4] Kuhn 1962.

$$\left\{ \begin{array}{ccc} \boldsymbol{C} & B & \boldsymbol{A} \\ \Downarrow & \Downarrow & \Downarrow \\ \boldsymbol{E} & D & \boldsymbol{C} \end{array} \right\} \Rrightarrow \left\{ \begin{array}{ccc} \boldsymbol{A} & B & \boldsymbol{C} \\ \Downarrow & \Downarrow & \Downarrow \\ \boldsymbol{E} & D & \boldsymbol{C} \end{array} \right\}$$

Figure 2 Bijectivity preserving construction of the $\boldsymbol{C \Rightarrow C}$ identity subbijection within the bijection by simply *switching* the pre-image elements C and A

Formalizing this simple common element removal operation as a theorem

We now formalize the above common element removal operation as a theorem, noting that it can be considered an elementary "generalized permutation" of a bijection or a combination of such. It is important to note that the now standard concept of a permutation as an arbitrary bijection from a set onto itself can easily and fruitfully be generalized, extended, and applied to bijections in general. However, even a basic study of these generalizations/extensions is beyond the scope of this paper.

THEOREM: Given a bijection $B(SP,SI)$ from a pre-image set SP onto an image set SI, where SP and SI have at least one element EC in common, then using only simple bijectivity preserving operations one can construct a bijection B^* from $SP-\{EC\}$ onto $SI-\{EC\}$, i.e. $B^*(SP-\{EC\},SI-\{EC\})$.

PROOF : In constructing this new bijection B^* we have only 2 possible cases for the common element EC:

1) If the common element EC is already paired with itself (subbijected onto itself under the bijection), then we can entirely remove this identity pairing (the identity subbijection of the pre-image EC onto its image self), and what remains will trivially be a bijection from SP-$\{EC\}$ onto SI-$\{EC\}$, the new bijection $B^*(SP\text{-}\{EC\},SI\text{-}\{EC\})$.

$$\left\{ \begin{array}{cccc} EP_1 & \boldsymbol{EC} & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ EI_1 & \boldsymbol{EC} & EI_3 & \dots \end{array} \right\} \Rrightarrow \left\{ \begin{array}{ccc} EP_1 & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow \\ EI_1 & EI_3 & \dots \end{array} \right\}$$

Figure 3 Bijectivity preserving elimination of the EC identity subbijection

Bijectivity is trivially preserved by this operation. (The removal of a subbijection is an example of generalizing and extending the standard concept of "permutation" as the bijection of a set onto itself to more general bijectivity preserving operations on bijections.) In particular, we need not "reorder" (a la Cantor) any elements of SP or SP-$\{EC\}$ with respect to SI or SI-$\{EC\}$.

2) If EC is not identity paired with itself (subbijected onto its image self under $B(SP,SI)$), then EC in SP must be paired with some element EI in SI and some element EP in SP must be paired with EC in SI. In a (trivially) bijectivity preserving fashion, we can switch the pre-image elements EC and EP (a standard permutation, thought of as an operation, except that the pre-image and image sets are here not in general the same).

$$\left\{\begin{matrix} EP_3 & EP_2 & \mathbf{EC} & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \mathbf{EC} & EI_2 & EI_3 & \dots \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathbf{EC} & EP_2 & EP_3 & \dots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \mathbf{EC} & EP_2 & EI_3 & \dots \end{matrix}\right\}$$

Figure 4 Bijectivity preserving construction of the EC identity subbijection within the bijection by simply *switching* pre-image elements

We now have a generalized and bijectivity preserving "permutation" of B, with the common element EC identity paired with its image self, the pre-image EP paired with (subbijected onto) the image EI, and the rest of the bijection, i.e. from $SP\text{-}\{EC,EP\}$ onto $SI\text{-}\{EC,EI\}$, remaining the same as in the original bijection. As in case 1), the identity pairing/subbijection from EC onto itself can be removed, leaving the (sub-) bijection from $SP\text{-}\{EC\}$ onto $SI\text{-}\{EC\}$, the needed bijection $B^*(SP\text{-}\{EC\},SI\text{-}\{EC\})$. □

This quasi-formally described operation and its result can easily be translated into a formally rigorous theorem. The operations involved cannot "generalizedly permute" a valid bijection into a non-bijection. We should also note the corollary, that if one derives/obtains a non-bijection as a result of applying this operation—or any sequence of such operations—to an ostensibly valid bijective mapping, that initial mapping cannot have been a valid bijection.

Dedekind and his Trojan Horse

Incomprehensibly, this at first glance innocuously correct theorem and proof, both simple and obvious once pointed out, has never been published in any mathematical work in the last two centuries, especially neither by Dedekind nor Cantor, and, far less understandably, neither by J. Henri Poincaré (1854-1912), considered one of the five greatest mathematicians in history. We can think of Dedekind's concept of Dedekind-infinite, i.e. of a transfinite set defined as one that can be bijected onto a proper subset of itself, as a Trojan Horse, never really "looked into", never really "vetted", yet "infinitely" desirable to quickly bring within the city gates because it neatly formally summarized the paradoxes held so long to be so inherent in infinity that they necessarily defined infinity. (This sense of definition was later modified theoretically, but that does not affect our investigations here.)

Just as the Trojans should have "vetted" the Trojan Horse designed by the incomparably crafty Greek Odysseus and left for the Trojans in hopes that they would take it within the city walls, Dedekind, Cantor, et al, should have carefully vetted this concept now known as Dedekind-infinite before it was made fundamental within the newly evolving set theory of Georg Cantor. They should have rigorously tested Dedekind's formalization using the newly developing formal tools of set theory to study the question of whether the paradoxes of infinity were mere ancient mathematical naiveté, or whether their formalization would have the necessary mathematical consistency when made essential to the foundations of mathematics.

It is inside this Trojan Horse of Dedekind that, instead of Greeks, we find the Evil Twin of the Banach-Tarski Paradox. Instead of a solid ball that can be doubled, just as infinity can be doubled and still equal infinity, we find the other half of Banach-Tarski hiding in this cornerstone of set theory, hiding within the concept of Dedekind-infinite. We find:

- **"Paradoxical Bijections"**
  Applying the simple bijectivity preserving operation of the equally simple theorem proven above to *all* the common elements of the pre-image and image sets of Dedekind-infinite bijections gives us "Paradoxical Bijections", bijections from non-empty sets onto the empty set, corresponding to "subtracting" a solid ball from both halves (reversed) of the implied equation (the rearrangement is reversible) of Figure 0 to get the result that 1 solid ball equals… nothing.

Theoretical difficulties

We find theoretical difficulties here of the utmost formal importance. The reader will remember that the formal definition of mathematical "theory" is given as the basic assumptions—the axioms and rules of inference—of the theory, together with *all* the theorems that can even *possibly* be derived from them. There is no discrimination at this stage—or at any later stage—based on whether or not the theorem that has just been proven contradicts a previous theorem.

The royal tradition of primogeniture—which here would be that the first theorem proven takes precedence over any later proven contradictory theorem, which latter is therefore not accepted into the royal line—has no *formal* place in a mathematical theory. The same holds for the royal tradition of *agnatic* primogeniture—which here would be that a second, "more royal" theorem proven later takes precedence over the earlier, now the "contradictory" theorem, which previously former now latter is (almost Biblically) therefore disowned by/-from the royal line. This too has no *formal* place in a mathematical theory, though we may discover informal/extra-formal occurrences.

And, the formal definition of an "inconsistent (mathematical) theory" is that the theory is inconsistent if it is even *possible* to derive contradictory theorems *within* the theory. So, because of the formal definitions of "theory" and "inconsistent theory", one may *never* theoretically reject any derivation (*within* the theory) as "invalid" merely because it results in a contradiction. Likewise, the derivation of a contradiction may *never* be taken as indicating that the derivation took place outside of the theory in question.

If one is *not* able to remove, as described above, *all* the common elements of a given Dedekind-infinite bijection, then set theoretically there must exist *within* set theory the contradiction of the simultaneous existence of a bijection where both the pre-image and image sets have a common element, and that that same element both *can* and *cannot* be removed using the simple theorem-based bijectivity preserving operation proven above—within set theory. So, if we insist on the formally defined consistency of set theory, then *within* set theory we must be validly able to derive—and therefore by the formal definition of "theory" must derive—"Paradoxical Bijections", the Evil Twin of the Banach-Tarski Paradox!

To make that result even more compelling, we will also note that if $\mathbb{N}\equiv\{1,2,3,...\}$ is the image set of a Dedekind-infinite bijection, and the pre-image set is therefore a proper superset of $\mathbb{N}$, we can use finite induction, often known as mathematical induction, to remove *all* the natural numbers from the pre-image and image sets[5], giving us as before a "Paradoxical Bijection" from a non-empty set onto the empty set. The reason this argument can be considered more compelling is that to deny this argument we would have to abandon finite induction. Since finite induction is the conjoined identical twin of the Axiom of Infinity, abandoning finite induction is tantamount to abandoning and thus *falsifying* the Axiom of Infinity… and much of set theory along with it.

It will be important to some to here also note the relationship to set subtraction of applying the common element removal theorem/operation given above to remove all common elements in a Dedekind-infinite bijection. There is a common complaint/ objection/comment from beginning mathematicians and non-mathematicians concerning the disparity between the set subtraction from a Dedekind-infinite set of its defining proper subset and the standard transfinite cardinality result that $\aleph_0 + n = \aleph_0$ [6], which derives from the cardinalities of a Dedekind-infinite set that has a finite number of elements "more" than its proper subset: "But there are still elements left over!"

Re-evaluating a paradigmatic construction of a Dedekind-infinite bijection

Look at Figure 5 and imagine that we have a bijection consisting of *any* number of pre-image ●s each subbijected onto a single image U. The reason for using all ●s to represent the pre-image elements is that Cantor's concept of cardinals is that, not only were any orderings of the elements abstracted out (part of Cantor's abstracting cardinals from ordinals), but even what the elements were, e.g. the values of any numbers and even the fact that they were numbers, was/were abstracted out. Only the theoretical fact that the elements were distinct from each other was not abstracted out. The image elements are represented as Us for similar reasons, but also to distinguish them from the pre-image elements. So, e.g. the numbers 2 and 32 are each distinct single elements, cardinally, and can be represented for our cardinal purposes here as ●s or Us).

Now look at Figure 6 and notice the as yet unsubbijected pre-image element ● on the far left. Try to decide which of all the subbijected ●s you want to switch that unsubbijected ● with in order to subbiject onto its U in the final total bijection of all the ●s onto all the Us. Remember that we eventually need to subbiject *all* the pre-image ●s onto all the image Us

[5] Borowski and Borwein 1991, 222.
[6] Cantor 1915, §6, (2), 104.

without adding an (unsubbijected) U… and *without* "disappearing" a ●. Any unsubbijected pre-image ● must eventually become subbijected onto some image U before we can complete the construction of this final total bijection that set theory says we *are* theoretically able to construct (and therefore theoretically *must* construct as part of deriving all possible theorems within the theory). But we tend to forget that every U we might think to add to gain a free U must necessarily come un-free, with a ● already subbijected onto it.

{ ● ● ● ● ● ● ● ● ● ● ● …
⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓
U U U U U U U U U U U … }

Figure 5 We have a bijection with any number of single pre-image element to single image element subbijections, i.e. any number of ● ⇒U subbijections

{ ● ● ● ● ● ● ● ● ● ● ● …
‡ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓
U U U U U U U U U U … }

Figure 6 Which subbijected ● do you want to switch the unsubbijected ● (on the far left) with to start constructing the total bijection?!

Proceeding in this simple fashion makes the situation seem less… uncomplicated than it did as when proceeding unquestioningly as Cantor did over a century ago, which everyone accepted happily… almost everyone. Indeed, we have over a century of making progress in the direction pointed out by Cantor, Dedekind, Hilbert, Russell, Banach, Tarski, Gödel, Cohen, et al. A strong feeling… escalates that something has been overlooked somewhere.

Did Cantor, et al, overlook anything?!

Let's backtrack a bit. The equinumerosity of the natural numbers and the even natural numbers has been considered obvious for millennia. The usual way of proceeding to show this is to merely point out that $n \Rightarrow 2n$ for all natural numbers n. Cantor thought of infinity, or rather the first (and the only countable) cardinal infinity $\aleph_0$, as being the first (cardinal) number that could not be made larger by adding 1. His definition was considered more naïve than that of Dedekind, even though it was logically equivalent, and it never came to be accepted as ready for prime time. However, it did form the basis for Cantor's demonstration of his concept of the "reordering" of a set, i.e. of a set that *seemed* to be larger than e.g. $\mathbb{N} \equiv \{1,2,3,...\}$ of cardinality $\aleph_0$, so as to put it into a strict one-to-one correspondence with (or bijection onto) $\mathbb{N}$ itself. This "reordering" does not have to be of an ordered set, much like the modern definition of a permutation as a bijection of a potentially unordered set onto itself. And such a "reordering" is happening implicitly in the $n \Rightarrow 2n$ mapping above that seems bijective. It is this "reordering" concept of Cantor's that the bijectivity preserving common element removal results shown earlier suggest that we look at more closely for… "oversights", which it turns out are easy to find.

Cantor gave the example (or something close to it) of "reordering" the set $\mathbb{N}\cup\{0\}$ (a "proper superset" of $\mathbb{N}$) so as to put it into a one-to-one correspondence with $\mathbb{N}$. He mapped, seemingly bijectively, every n in $\mathbb{N}\cup\{0\}$ onto $n+1$ in $\mathbb{N}$. This operation embodied his concept of "reordering" the elements of $\mathbb{N}\cup\{0\}$ so as to put them all into a strict one-to-one correspondence with the elements of $\mathbb{N}$, and his concept that the first cardinal infinity, or for that matter any cardinal infinity, could not be made larger by adding 1. Given the seeming success of this "reordering", we can see how the transfinite arithmetic result, $\aleph_0+1=\aleph_0$, immediately follows. And from $\aleph_0+1=\aleph_0$ follow $\aleph_0=2\cdot\aleph_0$ and $2^{\aleph_0}=2\cdot 2^{\aleph_0}$, the well-known transfinite arithmetic results that so resemble the Banach-Tarski Paradox, and that can be seen to derive initially from Dedekind-infinite bijections.

But Cantor seems to have forgotten a fundamental principle of mathematics, that when one defines an entity, and then uses that defined entity, one must always be able to re-substitute the definition for the defined entity everywhere it is used and get the same result. In our case, Cantor seemingly performed an infinite number of bijective mappings, $n\Rightarrow n+1$, "simultaneously". At least no one was concerned about whether these mappings were "simultaneous" or not.

But, by one of the most important fundamental principles in mathematics, we must be able to do this infinite number of $n\Rightarrow n+1$ mappings precisely as infinity is defined in the Axiom of Infinity. There, infinity, or rather the infinite set of all natural numbers $\mathbb{N}\equiv\{1,2,3,...\}$, is first partially defined as containing the (starting) element 1; then, a conditional says that if n is an element of $\mathbb{N}$, then $n+1$ is also an element of $\mathbb{N}$; so, we start with 1 and continue with 2, then 3… until we have used up all the natural numbers, precisely as in a proof by finite induction, or in an infinite schema with $\mathbb{N}$ as its index set.

For Figure 6, this means that in order to construct the total bijection from the set of all ●s (including the unsubbijected ●) onto the set of all Us, we either start with the subbijected ● in the subbijection ●⇒U corresponding with the natural number 1. (Or alternatively we could start with the ● in the ●⇒U subbijection that is indexed by the natural number 1 in the index set $\mathbb{N}$.) We see the result of this in Figure 7. It should look familiar.

{ ● ● ● ● ● ● ● ● ● ● ● …
‡ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓ ⇓
U U U U U U U U U U … }

Figure 7 We have switched the unsubbijected ● with the ● in the first ●⇒U subbijection, corresponding to the natural number 1. What now?!

When we use the term "logically equivalent", it is essential to note that its use must be in a specific context, for a specific purpose. This is much like it is with the fallacies of logic set out by Aristotle millennia ago: the entities reasoned about must retain their precise values throughout the reasoning process, which someone simplified to the requirement that, for any entity A that is involved in a specific reasoning process, it is essential that $A=A$ throughout that process. If we have ● and ● involved in a process of reasoning, it may be that they are not logically equivalent since we may be referring to their precise position, which is not the same. So, the context and purpose are essential to the use of the term "logically equivalent".

In the case of Figure 6 and Figure 7, and our specific context-purpose of constructing a total bijection from the set of all subbijected and unsubbijected ●s onto the set of all (subbijected) ⋃s, we can immediately notice that switching the unsubbijected ● with the first subbijected ● is logically equivalent to not switching it/them. It is also logically equivalent to switching them *any* number of times, even an infinite number of times, even Cantor's absolute infinite (number) of times. We can also immediately notice that, in this same context and with this same purpose, each and every ●⇒⋃ subbijection is logically equivalent to every other. We get the same result if we switch the unsubbijected ● with the first, the second, or any other subbijected ●. We can also (less immediately) notice that it doesn't really matter how many ●⇒⋃ subbijections we have, since they are all logically equivalent here, and we cannot really get any further than Figure 6 and Figure 7 suggest that we can in constructing the total bijection that we need; we could have *any* number of ●⇒⋃ subbijections, even Cantor's absolute infinite (number), even none, and we would not be able to construct the total bijection we need.

So, if we insist on applying the principle that we must be able to replace each and every defined entity with its original definition and still get the same result, we find that Cantor's construction of a simple Dedekind-infinite bijection using his process of "reordering" cannot be made to even appear to work at all. In fact, it starts to be "obvious" that the "cardinality" (now unexpectedly finding itself in question) of every set will be increased by adding a new element. This leads to many… consequences.

What Evils of this Banach-Tarski Twin lie in wait for us?!

Above we discovered the Evil Twin of the Banach-Tarski Paradox, "Paradoxical Bijections" from non-empty sets onto the empty set that can be readily and with formal validity derived within set theory. Accepting this Evil twin of "Paradoxical Bijections" may turn out to be decidedly more difficult than accepting the Banach-Tarski Paradox has been. But if we do not formally-theoretically accept this Evil Twin as we have formally-theoretically accepted Banach-Tarski, we unavoidably face the necessity of starting to seriously question set theory for the first time in over a century. We find ourselves in a traditional double bind, impaled on the horns of a theoretical *and* moral dilemma, faced with a Hobson's choice, and all of these together with "zugzwang"[7]:

1) **Either** we accept **"Paradoxical Bijections"** as the—co-equal—**"Evil Twin of the Banach-Tarski Paradox"**; or…

[7] "Zugzwang" is German for "compulsion to move", and is most often used when the only legal moves a chess player has are disadvantageous, and of course not moving is not a legal option either.

2) **Or**, as a community, we must commence a serious and sincere re-examination of set theory, we could even say an inquest, or more modernly we could conceive of it as a **"deconstruction"[8] of set theory**.

One of the most salient issues, and difficult to assimilate, about the above simple bijection "permutation" theorem, its application to Dedekind-infinite bijections, and the discovery of a significant flaw in Cantor's construction of a Dedekind-infinite set is that they seem to provide overwhelming evidence that the cardinality of any set is made larger by adding a new element. This would have "serious consequences" for set theory.

If Banach-Tarski's Evil Twin has its way, set theory will lose one of its favorite offspring:

1) the **Continuum Hypothesis (CH)**.

This loss is inevitable if the cardinality of any set increases when a new element is added. We would need to change $\aleph_0 + 1 = \aleph_0$ to $``\aleph_0" + 1 > ``\aleph_0"$. The ""s around $``\aleph_0"$ are to indicate that:

2) $``\aleph_0"$, the first transfinite cardinal, and cardinal transfinite arithmetic in general,

would also be casualties as their definitions depend on the theoretical existence of Dedekind-infinite sets, in particular:

3) $\mathbb{N} \equiv$ **{1,2,3,...}**, as the ***set*** of all natural numbers, the **Axiom of Infinity** that makes it a set,

are yet more casualties as $\mathbb{N}$ could not be a theoretically proper set. Even the "cardinality" of Cantor's "absolute infinite" must be increased by adding a new element, but in any case Cantor held it to be inherently inconsistent, because for him it represented (*was*) God[9], who is beyond any possible human sense of "consistency".

It was mentioned that $\mathbb{N}$ would be a casualty. This is better understood if we understand that:

4) our concept of **"cardinals"** in one way they are distinguished from "ordinals"

would also be a casualty, i.e. in the sense that the (finite or transfinite) ordinals keep increasing successively by "1" up to Cantor's "absolute infinite". The finite cardinals would no longer have $``\aleph_0"$ as a "ceiling" to "stop" at; the transfinite cardinals would not escape increase by adding 1. We would likewise even lose our concept of:

5) the **"cardinality of the Continuum"**.

(Our concept of the Continuum in set theory is so important that capitalizing the word, at least in this situation, seems only fitting.) And, more subtly and far more importantly, we would lose:

6) **the Continuum**, because we lose our heretofore unquestioned concept that **the Continuum** can be *constructed* from points or sets of points!

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[8] Only crudely in the sense of French philosopher, Jacques Derrida, with apologies. Wikipedia, "Jacques Derrida"

[9] Dauben 1979.

What *can* be constructed from points and/or sets of points will come to be called a "quantinuum", or since the multiplicity of such entities is obviously boundless, e.g. the prime decompositions of their bases (reals as infinite base expansions) would most likely be different, even incommensurable, the plural "quantinua" would seem more to the point. Points and/or sets of points will be embeddable in a "true continuum", or in "true continua", but they will never be able to properly define it or construct it. And to round out this apocalypse, we would also lose:

7) **real numbers**, as we conceive of them, since our concept of infinite decimal expansion real numbers with $\aleph_0$ decimal places to the right of the decimal point falters.[10]

Summary

> *"When a long established system is attacked, it usually happens that the attack begins only at a single point, where the weakness of the doctrine is peculiarly evident. But criticism, when once invited, is apt to extend much further than the most daring, at first, would have wished."*
> *Bertrand Russell*[11]

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[10] Knowles 2004.

[11] Russell 1897, 1996, Chap. I, 17.