Does the Banach-Tarski Paradox have an *Evil Twin?!*

Paradoxical Bijections

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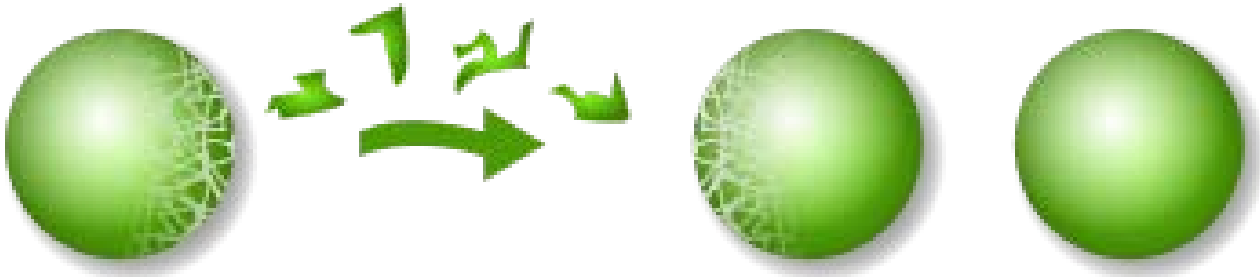
Audience Advisory
Part 1

- This presentation (1079-03-89) at the 2012 Spring Southeastern Sectional Meeting (#1079) of the American Mathematical Society at the University of South Florida in Tampa, FL, on Saturday, March 10, and its accompanying paper are **intended for a general audience, familiar with basic transfinite Set Theory**.

Audience Advisory
Part 2

- It will help very much if you know in advance roughly what the
- **Banach-Tarski Paradox** is, what
- **Dedekind-infinite sets** are, what
- **bijections** are,
- and that they are all **essential to Set Theory**.

The Banach-Tarski Paradox



- Accepted theorem in set theoretic geometry:

  a finite number of "strange pieces" of
  1 solid 3-D ball are rearranged into
  2 solid 3-D balls the same size

- (Artwork sans permission from http://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox.)

The Simple Origins of The *Evil Twin* of Banach-Tarski

- **… but first some definitions and history**
- **Definition:** a “**Theory**” is modernly defined as **the basic assumptions, i.e. the axioms and rules of inference, together with *all* the theorems that can *even possibly* be derived from them.**

Inconsistency

- **Definition:** a theory is considered to be "**inconsistent**" if it is *even **possible*** to derive a **contradiction** from the axioms and rules of inference.
- A **corollary** is that one may **never dismiss** a derivation as invalid just because it results in a **contradiction**.

Bijections

- **Definition:** a "**Bijection**" from a **pre-image set** onto an **image set** is the modern formal term for the **strict one-to-one correspondence(s)** between the elements of those sets.

  This is one of the most fundamental concepts in Set Theory, a **sine qua non**.

Dedekind-infinite sets Part 1

- **Definition:** a "**<u>Dedekind-infinite set</u>**" is a set that can be put into a **strict one-to-one correspondence with a proper subset** of itself.

  E.g. **the positive integers** with **the even positive integers.**

Dedekind-infinite sets
Part 2

- Also e.g. Cantor **"reordered"** the **set** $\mathbb{N}\cup$**{0}={0,1,2,3,…}**

  to put it into a

  strict one-to-one correspondence with the **set** $\mathbb{N}\equiv$**{1,2,3,…}**

  by mapping, seemingly bijectively, **every** ***n*** in $\mathbb{N}\cup$**{0}** onto ***n*****+1** in $\mathbb{N}$.

Dedekind-infinite sets
Part 3

- **Dedekind's concept is the best summary mathematical formalization of the paradoxes of infinity ever developed.** This is essential because everyone had/has come to believe that

<u>paradox is inherent in infinity</u>.

- Dedekind was the first to add the idea that **infinity is *defined* by its inherent paradoxes**.

Dedekind-infinite sets
Part 4

- Cantor had the logically equivalent concept of (cardinal) infinity as **a (cardinal) number that cannot be made larger by adding 1**.
- This was considered by many to be too naïve, “not ready for prime time”.
- But, from it was derived the **first equation in Cantor’s transfinite cardinal arithmetic**:

$$\aleph_0 + 1 = \aleph_0$$

Banach-Tarski and Transfinite Arithmetic

- From $\aleph_0 + 1 = \aleph_0$ Cantor derived many other transfinite arithmetic results, including:

$$\aleph_0 = 2 \cdot \aleph_0 \quad \text{and} \quad 2^{\aleph_0} = 2 \cdot 2^{\aleph_0}$$

- Note the interesting relationship between these results and **Banach-Tarski**: the appearance that, paradoxically, **"1 = 2"**, i.e. that **1 ball = 2 balls**.

Dedekind-infinite sets
Part 5

- Set theory embraced Dedekind's concept… without ever **"vetting"** this **"Trojan Horse"** for **"Greeks"**.
- The question of whether the **ancient paradoxes "inherent in infinity"** were **mere ancient naiveté** was never answered by mathematicians… nor even truly asked.

The Axiom of Infinity

- The simplest form of the **Axiom of Infinity** (**AI**) is the oldest:

1) **1** is a **member** of the **set** of **all natural numbers** $\mathbb{N}$;

2) **if** ***n*** is a **member** of $\mathbb{N}$, **then** ***n*+1** is also a member of the **set** $\mathbb{N}$;

together these **define** $\mathbb{N} \equiv$ **{1,2,3,...}**.

Finite Induction
Part 1

- **Finite Induction (FI) is the "conjoined identical twin" of the Axiom of Infinity.**
- They **both start** with **1**, and they **both have** that the case for ***n*** **implies** the case for ***n*+1**.

Finite Induction Part 2

- In the **Axiom of Infinity**, the **predicate** to be **“proven”** for **each & every natural number** is its **membership in the set of all natural numbers,** $\mathbb{N}$.

Finite Induction
Part 3

- In **Finite Induction**, the **predicate** to be proven is chosen at the time of the Finite Induction based proof.
- But *both* **AI** and **FI** construct or prove the **predicate** for ***all*** **natural numbers**.
- See Borowski and Borwein, *The HarperCollins Dictionary of Mathematics*, New York: HarperCollins, 1991, p. 222.

“Dedekind-infinite Bijections”

- **Non-standard Definition:**

  A **“Dedekind-infinite Bijection”** is any **bijection** from a set onto a proper subset of itself **that demonstrates** that the first set is **Dedekind-infinite**.

The Simple Origins of The *Evil Twin* of Banach-Tarski

- **New: a simple bijectivity preserving operation performed on bijections**,

a generalized **“permutation” of an arbitrary *bijection***, has been found that, when applied to a bijection, removes an **arbitrary element** that is **common** to both the pre-image and image sets (if such an element exists).

It has consequences…

Simple "Permutation" of a Bijection Part 1

{	*A*	*B*	***C***	}		{	*A*	*B*	}
	$\Downarrow$	$\Downarrow$	$\Downarrow$		$\Rrightarrow$		$\Downarrow$	$\Downarrow$	
{	*E*	*D*	***C***	}		{	*E*	*D*	}

Figure 1 Simple bijectivity preserving "permutation" of the *bijection*, eliminating the identity subbijection, $\boldsymbol{C} \Rightarrow \boldsymbol{C}$, constructing the bijection with common element removed.

Simple "Permutation" of a Bijection Part 2

{	*C*	*B*	*A*	}		{	*A*	*B*	*C*	}
	$\Downarrow$	$\Downarrow$	$\Downarrow$		$\Rrightarrow$		$\Downarrow$	$\Downarrow$	$\Downarrow$	
{	*E*	*D*	*C*	}		{	*E*	*D*	*C*	}

Figure 2 Bijectivity preserving construction of the identity subbijection, $\boldsymbol{C} \Rightarrow \boldsymbol{C}$, by simply *switching* elements ***C*** and ***A*** in the pre-image set, an additional generalized "permutation" of the *bijection*.

Simple "Permutation" of a Bijection
Part 3

$\{$	A	B	$\boldsymbol{C}$	$\}$		$\{$	A	B	$\}$
	$\Downarrow$	$\Downarrow$	$\Downarrow$		$\Rrightarrow$		$\Downarrow$	$\Downarrow$	
$\{$	E	D	$\boldsymbol{C}$	$\}$		$\{$	E	D	$\}$

Figure 3 We repeat the simple bijectivity preserving "permutation" of the $\boldsymbol{C} \Rightarrow \boldsymbol{C}$ identity subbijection, again constructing the bijection with common element removed.

Simple "Permutation" of a Bijection Part 4

- This simple **bijection "permutation"** can be formalized in a simple theorem:

  Given a bijection ***B*(*SP*,*SI*)** from a pre-image set ***SP*** onto an image set ***SI***, where ***SP*** and ***SI*** have at least one element ***EC*** in common, then using only simple **bijectivity preserving operations** one can **construct a bijection** ***B**** from ***SP*–{*EC*}** onto ***SI*–{*EC*}**, i.e.

  ***B**(*SP*–{*EC*},*SI*–{*EC*})**.

Compound "Permutation" of a Bijection Part 1

- **An obvious compound "permutation" of an arbitrary bijection is to remove *all* of its common elements, while completely preserving bijectivity.**

- **This could easily be made a theorem.**

Compound "Permutation" of a Bijection Part 2

- If for some reason we are unable to thus remove *all* of its common elements, **then there must exist a common element that cannot be thus removed**, **contradicting the *simple* bijection "permutation" theorem** given above.

Compound "Permutation" of a Bijection Part 3

- If the **set** of *all* (and *only*) the **common elements** happens to be $\mathbb{N}$,

  Finite Induction guarantees that *all* common elements can be removed, with bijectivity completely preserved.

Compound "Permutation" of a Bijection Part 4

- **If** for any reason we are unable to thus use **Finite Induction** to **bijectivity preservingly remove *all* the common elements** (*all* in $\mathbb{N}$), **then** we find ourselves ***falsifying*** not only **Finite Induction**, but **also** its conjoined identical twin, the **Axiom of Infinity**.

But… what about "Dedekind-infinite Bijections"?!

- **The *Evil Twin* of the Banach-Tarski Paradox** now makes itself known:

- **Preserving bijectivity completely, we remove *all* the common elements from both the pre-image and image sets of "Dedekind-infinite Bijections", and *within* Set Theory we derive…**

Banach-Tarski's *Evil Twin!*

"<u>Paradoxical Bijections</u>"

from <u>*non-empty* sets</u>

onto <u>the empty set</u>, Φ.

“But… *what about Cantor’s proof?!*”

- Cantor **“reordered”** $\mathbb{N}\cup$**{0}={0,1,2,3,…}** so as to put it into a **strict one-to-one correspondence** with $\mathbb{N}\equiv$**{1,2,3,…}** by **mapping**, seemingly bijectively, **every** ***n*** in $\mathbb{N}\cup$**{0}** onto ***n*+1** in $\mathbb{N}$.

Is there a *flaw* in Cantor's proof?! Part 1

- But now consider a ***completely cardinalized* bijection of any size** from a pre-image set consisting of ●s onto an image set consisting of Us:

● ● ● ● ●…

U U U U U…

We need to add a new ● to the pre-image ●s.

Is there a *flaw* in Cantor's proof?! Part 2

- Notice that there is **no free image** U to subbiject our unsubbijected pre-image ● onto.

●　　●　●　●　●　●...

　　U　U　U　U　U...

- In order to subbiject our unsubbijected pre-image ● onto a *chosen* subbijected image U, we need to desubbiject that subbijected ● and subbiject the previously unsubbijected ●.

Is there a *flaw* in Cantor's proof?! Part 3

- The *flaw* in Cantor's proof is already evident. We merely exchange

● ● ● ● ● ●…

⋓ ⋓ ⋓ ⋓ ⋓…

● ● ● ● ● ●… for

∪ ∪ ∪ ∪ ∪…

- I.e. it doesn't matter which subbijected ● we choose, we merely perform a **null operation**.

Is there a *flaw* in Cantor's proof?! Part 4

- **A fundamental principle of mathematics**, which Cantor ignored, says that one must be able to **substitute the initial definition for *every* instance of a defined entity** and obtain the ***same* result**.

- Here one **cannot obtain the same result**.

Is there a *flaw* in Cantor's proof?! Part 5

- Cantor conceived of his **paradigmatic and ostensibly bijective mapping** of
  each & every *n* in $\mathbb{N}\cup$**{0}={0,1,2,3,...}**
  onto
  each & every *n*+1 in $\mathbb{N}\equiv$**{1,2,3,...}**
  as happening somehow **"simultaneously"**, or such a question never occurred to him.

Is there a *flaw* in Cantor's proof?! Part 6

- Cantor overlooked that the **infinity** $\mathbb{N}$ **was *defined/constructed sequentially***, by **starting** with **1**, then **proceeding sequentially-successively** with **2, 3,…,** and that his construction had to succeed even proceeding thus, sequentially-successively.
- As we saw in Part 3, **this cannot succeed**.

What *Evil* does this *Twin* **threaten?!** Part 1

- **Either**, in the same way that we have so far readily accepted **Banach-Tarski**, we must equally accept its ***Evil Twin***, **“Paradoxical Bijections”,** or…

What *Evil* does this *Twin* **threaten?!** Part 2

- **Or**, as a community we must seriously start to **“vet” Dedekind’s “Trojan Horse”**, his concept of **“Dedekind-infinite”**, and anything else that seems suspiciously **“Greek**”, and *as a community* we must commence… **a “deconstruction” of Set Theory**.

What *Evil* does this *Twin* **threaten?!** Part 3

- If we choose to **"vet"**, one of the first things to notice is that the **simple bijection "permutation" theorem** given above combined with the *flaw* in Cantor's proof will mean that:

  adding a new element to any set constructs a *new* set that *always* has a greater cardinality.

- This immediately means that this *Evil Twin* **threatens** several **sine qua nons** of Set Theory:

What *Evil* does this *Twin* **threaten?!** Part 4

- It **threatens** that we must find that

$$\text{“}\aleph_0 + 1 > \aleph_0\text{”}$$

because the cardinality of

$\mathbb{N} \cup$ **{0} = {0,1,2,3,...}**

would necessarily exceed the cardinality of

$\mathbb{N} \equiv$ **{1,2,3,...}**

What *Evil* does this *Twin* **threaten?!** Part 5

- Thus it **threatens**:

The Continuum Hypothesis!

What *Evil* does this *Twin* **threaten?!** Part 6

- Further, it generally **threatens** our concepts of

1) $\mathbb{N} \equiv$ {1,2,3,...} and

2) the Axiom of Infinity that defines it,

3) the first transfinite cardinal, "$\aleph_0$",

4) transfinite cardinal arithmetic and

5) cardinality in general, especially

6) the cardinality of the Continuum, and

7) our overall "*idea*" of "The Continuum"…

What *Evil* does this *Twin* **threaten?!** Part 7

- and of course, it **threatens** our concept of

Real Numbers

- **So, on the horns of our theoretical *and* moral dilemma, our Hobson's Choice is…**

Banach-Tarski's *Evil Twin!*

- ***Accept*** **"Paradoxical Bijections"** just as we have accepted the **Banach-Tarski Paradox**, or…
- ***Or*****, commence a *community* "deconstruction" of Set Theory**.
- (With apologies to Jacques Derrida.)

Banach-Tarski's
Evil Twin!

- *"When a long established system is attacked, it usually happens that the attack begins only at a single point, where the weakness of the doctrine is peculiarly evident. But criticism, when once invited, is apt to extend much further than the most daring, at first, would have wished."*
- **Bertrand Russell, 1897, *An Essay on the Foundations of Geometry***
- A long term *community* "deconstruction" of Set Theory…